6. THE t'S IN TERMS OF THE s'S. (THE b'S IN TERMS OF THE s'S.) (Inverse of 5.)

Weight 1.

	s_1
t_1	1

Weight 2.

	s_1^2	s_2
t_2	$\frac{1}{2}$	$\frac{1}{2}$
t_1^2	1	

Weight 3.

	s_1^3	$s_1 s_2$	s_3
t_3	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$
$t_1 t_2$	$\frac{1}{2}$	$\frac{1}{2}$	
t_1^3	1		

Weight 4.

	s_1^4	$s_1^2 s_2$	s_2^2	$s_1 s_3$	s_4
t_4	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{4}$
$t_1 t_3$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$	
t_2^2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$		
$t_1^2 t_2$	$\frac{1}{2}$	$\frac{1}{2}$			
t_1^4	1				

Weight 5.

	s_1^5	$s_1^3 s_2$	$s_1 s_2^2$	$s_1^2 s_3$	$s_2 s_3$	$s_1 s_4$	s_5
t_5	$\frac{1}{120}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{5}$
$t_1 t_4$	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	0	$\frac{1}{4}$	
$t_2 t_3$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$		
$t_1^2 t_3$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$			
$t_1 t_2^2$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$				
$t_1^3 t_2$	$\frac{1}{2}$	$\frac{1}{2}$					
t_1^5	1						

The sum of the horizontally collinear elements is 1, as it should be by (41).

If s is changed into $-s$, *i. e.*, if the signs of the elements of the columns are or are not changed according as the product of the s's heading the column is of odd or of even degree, and the t's are replaced by b's, we have then the table for the b's expressed in terms of the s's. In this case the sum of horizontally collinear elements, except for the last line, is zero, by (42) and (43).

TWO INFINITE SYSTEMS OF GROUPS GENERATED BY TWO OPERATORS OF ORDER FOUR.

By PROFESSOR G. A. MILLER. Stanford University.

If a, b, c represent respectively the orders of two operators and that of their product, the group generated by these operators is completely defined by the numbers a, b, c only in the following cases: (1) When one of these three numbers is unity; (2) When two of them are equal to 2 and the third greater than 1; (3) When they have one of the following three sets of values: (2, 3, 3), (2, 3, 4), (2, 3, 5). The corresponding groups are respectively cyclic, dihedral rotation, tetrahedral, octahedral, and icosahedral.*

The abstract definitions and the laws of combination of these groups are so simple that they furnish very interesting examples in the logic of algebra. We proceed to consider two other very elementary systems of groups. Let s_1, s_2, represent two non-commutative operators of order four such that

*Burnside, Theory of groups of finite order, 1897, p. 291; Miller, *Bulletin of American Mathematical Society*, Vol. 7, 1901, p. 424; *American Journal of Mathematics*, Vol. 24, 1902, p. 96.

$$s_1^2\ s_2^2=1.$$

By multiplying on the right and the left by s_1 and s_2 respectively there results

$$s_1^3\ s_2^3=s_1\ s_2=s_1^{-1}\ s_2^{-1}=(s_2\ s_1)^{-1}.$$

Since $s_2^{-1}(s_2\ s_1)s_2=s_1\ s_2=(s_2\ s_1)^{-1}$, it follows that s_2 transforms $s_2\ s_1$ into its inverse. The order (g) of the group (G) generated by s_1 and s_2 is therefore equal to twice the order of s_1s_2 whenever the group (H_1) generated by s_1s_2 includes $s_1^2=s_2^2$. When this condition is not satisfied $g=4h$, h being the order of H_1.

In the former case G is composed of the cyclic group H_1 of order $g/2$, and $g/2$ operators of order 4 which have a common square.* Each of these operators of order 4 transforms each operator of H_1 into its inverse. As s_1 and s_2 can be so selected that $g/2$ is an arbitrary even number, this infinite system of groups is composed of just one group of each order $4k$, $k=2, 3, \ldots\ldots, \infty$. This system is interesting also because *it includes all possible non-cyclic groups of order p^m, p being a prime, which have only one subgroup of order p^s, $0<s<m$.*†

In the latter case G includes the direct product (H_2) of an operator of order two (s_1^2) and the cyclic group of order $g/4$ generated by s_1s_2. The remaining $g/2$ operators of G are of order four, have a common square, and transform each operator of H_2 into its inverse. This infinite system is composed of just one group of each order $4k$, $k=3, 4, \ldots\ldots, \infty$. The value $k=2$ is excluded by the fact that two non-commutative operators of order 4 cannot satisfy both the conditions

$$s_1^2\ s_2^2=1 \text{ and } (s_2s_1)^2=1.$$

In fact, it results from the former that $s_1^3\ s_2^3=s_1s_2$, and from the latter that $s_2s_1s_2s_1=1$, or that $s_2s_1=s_1^3\ s_2^3$. Hence $s_1s_2=s_2s_1$. When k is odd, H_2 is cyclic.

Every group generated by two non-commutative operators of order four which have a common square must belong to one of the given systems. Hence such a group may be completely defined by saying that it is generated by two non-commutative operators of order 4 which have a common square and whose product is of order $h>2$; when h is even it is necessary to add whether the product does or does not generate the square of one of the given generators.

*This statement follows directly from the fact that s_2 transforms each operator of H_1 into its inverse.
†Burnside, Theory of Groups, p. 75.